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LETTER TO THE EDITOR

Periodic and wave-like solutions in non-linear lattices

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Abstract. We consider a lattice system described by a set of non-linear differential equations, and we discuss how general lattice properties and/or symmetries can allow the reduction of this problem to a lower-dimensional one, and possibly grant the existence of periodic or wave-like solutions.

We consider a system of lattice (non-linear) differential equations, i.e.

$$\dot{\xi}_\alpha = f_\alpha(\lambda; \xi) \tag{1}$$

where α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, with $\alpha_i = 1, \ldots, m_i$, and each component is an s-dimensional real vector describing an 'elementary site' in the lattice; $\xi = \xi(t)$ is then an N-dimensional ($N = sm_1m_2 \ldots m_d$) real vector, and $f: \mathbb{R}^p \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be a non-linear sufficiently smooth (e.g. C^∞ or polynomial) function. Here, $\lambda \in \mathbb{R}^p$ is a real p-dimensional control parameter, introduced in order that the discussion should also include dependence on physical parameters, bifurcation problems, and/or possible phase transitions, and so on.

We want to discuss how general lattice properties and/or symmetries can allow the reduction of this problem to a lower-dimensional one, and possibly grant the existence of periodic or wave-like solutions.

For definiteness, we consider periodic lattices, i.e. $\alpha \in \mathbb{Z}^d$ but such that if $\alpha_{(k^*)} = (\alpha_1, \ldots, \alpha_k + m_k, \ldots, \alpha_d)$ then $\xi_{\alpha(k^*)} = \xi_\alpha$; however, our considerations would be easily extended to infinite lattices.

Let us first consider, for ease of notation, $d = 1$, i.e. a one-dimensional lattice, and let us impose periodic boundary conditions, say $\xi_{m+1} = \xi_1$ (so we deal with a chain of m sites, $N = sm$). Clearly, if the sites are equivalent, then f is 'equivariant' under the shift operator ($m \times m$ block matrix)

$$\hat{S} = \begin{bmatrix} 0 & 0 & & \cdots & & \mathrm{I} \\ \mathrm{I} & 0 & & \cdots & & 0 \\ 0 & \mathrm{I} & 0 & & \cdots & 0 \\ & & \ddots & & \ddots & \\ 0 & 0 & 0 & & \cdots & 0 \\ 0 & & \cdots & & \mathrm{I} & 0 \end{bmatrix} \tag{2}$$

where I is the s-dimensional identity matrix, i.e. it satisfies

$$\hat{S}f(\lambda; \xi) = f(\lambda; \hat{S}\xi) \tag{3}$$

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which means in this case that

$$f_\alpha(\lambda;\, \xi_1, \xi_2, \ldots, \xi_m) = f_{\alpha+1}(\lambda;\, \xi_m, \xi_1, \xi_2, \ldots, \xi_{m-1}) \tag{3'}$$

for each $\alpha = 1, \ldots, m$, with $f_{m+1} = f_1$.

We now generalise this isotropic case by assuming that there is a connection between the αth and the $(\alpha+1)$th site, described by a set of m real invertible $s \times s$ matrices $K_\alpha : R^s_{(\alpha)} \to R^s_{(\alpha+1)}$, such that $K_m K_{m-1} \ldots K_1 = \mathrm{I}$:

$$\begin{array}{ccccccccccccc} \bullet & \xrightarrow{K_1} & \bullet & \xrightarrow{K_2} & \bullet & \cdots & \bullet & \xrightarrow{K_\alpha} & \bullet & \cdots & \bullet & \xrightarrow{K_m} & \bullet \\ 1 & & 2 & & 3 & & \alpha & & \alpha+1 & & m & & (1) \end{array} \tag{4}$$

(with the periodicity condition at the ends, i.e. $K_m : R^s_{(m)} \to R^s_{(1)}$) in such a way that the global problem (1) obeys the following equivariance (or covariance) conditions

$$\hat{K} f(\lambda;\, \xi) = f(\lambda;\, \hat{K}\xi) \tag{5}$$

where

$$\hat{K} = \begin{bmatrix} 0 & 0 & & \ldots & & K_m \\ K_1 & 0 & & \ldots & & 0 \\ 0 & K_2 & 0 & & \ldots & 0 \\ & & \ddots & & \ddots & \\ 0 & 0 & 0 & & \ldots & 0 \\ 0 & & \ldots & & K_{m-1} & 0 \end{bmatrix}$$

which equivalently expresses, generalising (3′), the following relationship between adjacent sites:

$$K_\alpha f_\alpha(\lambda;\, \xi_1, \xi_2, \ldots, \xi_m) = f_{\alpha+1}(\lambda;\, K_m\xi_m, K_1\xi_1, K_2\xi_2, \ldots, K_{m-1}\xi_{m-1}) \tag{6'}$$

(with $f_{m+1} = f_1$).

It is now easy to see that, putting

$$\xi_2 = K_1\xi_1,\quad \xi_3 = K_2K_1\xi_1\ ,\ldots,\quad \xi_m = K_{m-1}\ldots K_1\xi_1 \tag{7}$$

all the m subsystems become equivalent to the unique s-dimensional problem concerning a single site

$$\dot{\xi}_1 = f_1(\lambda;\, \xi_1, K_1\xi_1, \ldots) \equiv \phi(\lambda;\, \xi_1). \tag{8}$$

Before discussing this reduced system, let us see how this procedure can be extended to multidimensional, $d > 1$, lattices. Consider first a planar rectangular lattice: the sites can be labelled by two indices $\xi_{\alpha\beta} = \xi_{\alpha\beta}(t)$ with $\alpha = 1, \ldots, m$; $\beta = 1, \ldots, n$, $N = smn$, and assume that there is a 'horizontal' connection $\hat{H}$ operating on each row in a similar way as in the linear case, and a 'vertical' connection $\hat{V}$ operating on each column. Precisely, in order to be as general as possible, we assume the existence of a net of linear transformations, operating according to the following scheme:

$$
\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\arrow[r] & \alpha, \beta-1 \arrow[r, "H^{(\alpha)}_{\beta-1}"] \arrow[d, "V^{(\beta-1)}_{\alpha}"'] & \alpha, \beta \arrow[r, "H^{(\alpha)}_{\beta}"] \arrow[d, "V^{(\beta)}_{\alpha}"] & {} \\
\arrow[r] & \alpha+1, \beta-1 \arrow[r, "H^{(\alpha+1)}_{\beta-1}"] \arrow[d] & \alpha+1, \beta \arrow[r] \arrow[d] & {} \\
 & {} & {} &
\end{tikzcd}
\tag{9}
$$

where each dot represents a s-dimensional site, each $H^{(\alpha)}_{\beta}$ and $V^{(\beta)}_{\alpha}$ are $s\times s$ real invertible matrices, and the periodicity conditions

$$\xi_{\alpha,n+1}=\xi_{\alpha 1} \quad (\alpha=1,\ldots,m) \qquad \xi_{m+1,\beta}=\xi_{1\beta} \quad (\beta=1,\ldots,n)$$

are assumed.

We also require

$$
\begin{aligned}
&H^{(\alpha)}_{n}H^{(\alpha)}_{n-1}\ldots H^{(\alpha)}_{1}=\mathrm{I} \qquad && \alpha=1,\ldots,m \\
&V^{(\beta)}_{m}V^{(\beta)}_{m-1}\ldots V^{(\beta)}_{1}=\mathrm{I} \qquad && \beta=1,\ldots,n
\end{aligned}
\tag{10}
$$

and the following 'compatibility conditions', emerging from (9):

$$V^{(\beta)}_{\alpha}H^{(\alpha)}_{\beta-1}=H^{(\alpha+1)}_{\beta-1}V^{(\beta-1)}_{\alpha}. \tag{11}$$

Writing the global N-dimensional system as a set of mn (s-dimensional) subsystems

$$\dot{\xi}_{\alpha\beta}=f_{\alpha\beta}(\lambda;\xi_{11},\ldots,\xi_{1n},\xi_{21},\ldots,\ldots,\xi_{mn}) \tag{12}$$

the equivariance under $\hat{H}$ and $\hat{V}$ is respectively (no sum over α, β)

$$
\begin{aligned}
&H^{(\alpha)}_{\beta}f_{\alpha\beta}(\lambda;\xi_{11},\ldots,\ldots,\xi_{mn}) \\
&\qquad =f_{\alpha,\beta+1}(\lambda;H^{(1)}_{n}\xi_{1n},H^{(1)}_{1}\xi_{11},\ldots,H^{(1)}_{n-1}\xi_{1,n-1}, \\
&\qquad\quad \times H^{(2)}_{n}\xi_{2n},\ldots,\ldots,H^{(m)}_{n}\xi_{mn},\ldots,H^{(m)}_{n-1}\xi_{m,n-1}) \\
&V^{(\beta)}_{\alpha}f_{\alpha\beta}(\lambda;\xi_{11},\ldots,\ldots,\xi_{mn}) \\
&\qquad =f_{\alpha+1,\beta}(\lambda;V^{(1)}_{m}\xi_{m1},V^{(2)}_{m}\xi_{m2},\ldots,V^{(1)}_{1}\xi_{11},\ldots,V^{(n)}_{m-1}\xi_{m-1,n})
\end{aligned}
\tag{13}
$$

or, in global notation,

$$\hat{H}f(\lambda;\xi)=f(\lambda;\hat{H}\xi) \qquad \hat{V}f(\lambda;\xi)=f(\lambda;\hat{V}\xi). \tag{14}$$

It is not difficult to extend this scheme to planar lattices with different lattice symmetry, e.g. for triangular lattices

$$
\begin{tikzcd}
 & \alpha, \beta-1 \arrow[rr, "H^{(\alpha)}_{\beta-1}"] \arrow[dl, "D^{(-)(\beta-1)}_{\alpha}"'] \arrow[dr, "D^{(+)(\beta-1)}_{\alpha}"] & & \alpha, \beta \arrow[dl, "D^{(-)(\beta)}_{\alpha}"'] \arrow[dr] & \\
\alpha+1, \beta-1 \arrow[rr, "H^{(\alpha+1)}_{\beta-1}"'] & & \alpha+1, \beta \arrow[rr] & & \alpha+1, \beta+1
\end{tikzcd}
$$

As in the rectangular lattice above, (9), the compatibility conditions to be assumed in this case are

$$H^{(\alpha+1)}_{\beta-1}D^{(-)(\beta-1)}_{\alpha}=D^{(-)(\beta)}_{\alpha}H^{(\alpha)}_{\beta-1}=D^{(+)(\beta-1)}_{\alpha}.$$

Similarly, the extension to three-dimensional lattices (crystals), or possibly even to higher-dimensional structures, is straightforward. As above, the various transformations are subjected to the condition that the product of transformations along different paths joining two sites must give the same result. Alternatively, this may be stated by imposing that the composition of a chain of transformations along any closed path, must give the identity operator I on the elementary site. It is also clear that it is sufficient to impose these conditions only on the elementary lattice structure.

In the following, for sake of simplicity, we shall refer to the case of planar rectangular lattices (9).

Even in this case, as for the one-dimensional lattice, one can easily reduce the original problem to an s-dimensional one. Let us state the result in a theorem.

Theorem 1. Let the global N-dimensional problem (1) satisfy the equivariance requirements (14) (or equivalently let (12) satisfy (13)), with conditions (10), (11). Then, putting for each $\alpha = 2, \ldots, m$, and $\beta = 2, \ldots, n$:

$$\xi_{\alpha\beta} = V_{\alpha-1}^{(\beta)} \ldots V_1^{(\beta)} H_{\beta-1}^{(1)} \ldots H_2^{(1)} H_1^{(1)} \xi_{11} \tag{15}$$

each subsystem becomes equivalent to the unique s-dimensional problem, referred to the elementary site,

$$\dot{\xi}_{11} = f_{11}(\lambda; \xi_{11}, H_1^{(1)}\xi_{11}, \ldots) \equiv \phi(\lambda; \xi_{11}) \qquad \phi: \mathbb{R}^p \times \mathbb{R}^s \to \mathbb{R}^s. \tag{16}$$

Let us stress the essential role played in this problem by the equivariance under $\hat{H}$, $\hat{V}$ (or $\hat{K}$ in the one-dimensional case), in particular for the possibility of reducing the dimension of the problem; in fact, once $\hat{H}$, $\hat{V}$, $\hat{K}$ are given, the subspace specified by (15), where solutions are to be found, is precisely the invariant subspace under the cyclic groups generated by these operators (cf also [1-3]).

Note also that (11) implies that

$$\hat{H}\hat{V} = \hat{V}\hat{H}$$

and in addition that one can arbitrarily choose $n-1$ matrices $H_\beta^{(\alpha)}$ for each α (the nth is fixed by (10)) and, e.g., $m-1$ in the first 'column' of matrices $V_\alpha^{(1)}$, the remainder being determined by (11).

Consider now the reduced problem (16). It is clear that if one is able to find a solution $\xi_0 = \xi_0(t)$ for it, the solution is extended by means of rule (15) to the whole lattice. An interesting case would be $s = 2$: then Poincaré-Bendixson theory [4], or standard Hopf bifurcation results [5] can assure the existence of a periodic solution. We shall consider this possibility in some detail later.

Another interesting situation occurs if the system is symmetric under some group G. Given a real s-dimensional representation T of G, we can consider the case

$$f_{\alpha\beta}(\lambda; T\xi_{11}, \xi_{12}, \ldots, T\xi_{mn}) = Tf_{\alpha\beta}(\lambda; \xi_{11}, \xi_{12}, \ldots, \xi_{mn}) \tag{17}$$

or equivalently

$$f(\lambda; \hat{T}\xi) = \hat{T}f(\lambda; \xi) \tag{17'}$$

where $\hat{T}$ is the direct sum of mn copies of the representation T. It is natural to assume in this case (see also [6], however), for each α, β, that

$$H_\beta^{(\alpha)} \in C(T) \qquad V_\alpha^{(\beta)} \in C(T) \tag{18}$$

where $C(T)$ is the set of matrices commuting with T; then, if T is absolutely irreducible, one has $H_\beta^{(\alpha)} = V_\alpha^{(\beta)} = \mathrm{I}$ and $\hat{H}$, $\hat{V}$ are simply the 'horizontal' and 'vertical' shifts in the lattice.

It can be noted that, assuming (18), it is sufficient to require that (17) is verified only for one site, e.g. for f_{11}; in fact, using (13, 15) one has, e.g., for f_{12}

$$\begin{aligned} f_{12}(\lambda;\, T\xi_{11}, \ldots, \ldots, T\xi_{mn}) &= H_1^{(1)} f_{11}(\lambda;\, H_n^{(1)})^{-1} T\xi_{1n}, \ldots, (H_{n-1}^{(m)})^{-1} T\xi_{m,n-1}) \\ &= H_1^{(1)} T f_{11}(\lambda;\, (H_n^{(1)})^{-1}\xi_{1n}, \ldots) = T f_{12}(\lambda;\, \xi_{11}, \ldots, \xi_{mn}) \end{aligned}$$

and similarly for any other site.

Again with (18), more interesting situations can appear if T is irreducible in the real sense but not in the complex one [7], e.g. if T is the standard two-dimensional representation of the rotation group SO(2), an example which is closely related to the above-mentioned Hopf bifurcation problem, and which we are going to discuss from now on.

Let us consider a planar lattice, with $s = 2$, which is symmetric under SO(2) according to (17), and denote by $\hat{S}_\mathrm{H}$, $\hat{S}_\mathrm{V}$ the horizontal and vertical shifts in the lattice, i.e. the transformations defined by

$$H_\beta^{(\alpha)} = \mathrm{I} \qquad V_\alpha^{(\beta)} = \mathrm{I} \qquad \forall \alpha, \beta.$$

We then have the following lemma.

Lemma 2. A planar lattice satisfying (17), where T is the standard real two-dimensional representation of SO(2), is equivariant (in the sense of (13), (14)) under $\hat{S}_\mathrm{H}$, $\hat{S}_\mathrm{V}$ if and only if it is equivariant under the transformations $\hat{H}_0$, $\hat{V}_0$ defined by

$$H_{0\beta}^{(\alpha)} = \mathcal{R}(2\pi/n) \qquad V_{0\alpha}^{(\beta)} = \mathcal{R}(2\pi/m) \qquad \forall \alpha, \beta$$

$\mathcal{R}(\theta)$ being the rotation through angle θ.

Proof. Using (13) with $H_\beta^{(\alpha)} = \mathrm{I}$, one has, e.g.,

$$f_{11}(\lambda;\, \xi_{11}, \ldots) = f_{12}(\lambda;\, \xi_{1n}, \ldots)$$

and, as $\mathcal{R}_n \equiv \mathcal{R}(2\pi/n) \in T$,

$$\mathcal{R}_n f_{11}(\lambda;\, \xi_{11}, \ldots) = f_{12}(\lambda;\, \mathcal{R}_n \xi_{1n}, \ldots)$$

and similarly for the other sites, which expresses the covariance under $\hat{H}_0$. Conversely, assuming (13) with $H_\beta^{(\alpha)} = \mathcal{R}_n$, one has

$$\mathcal{R}_n f_{11}(\lambda;\, \xi_{11}, \ldots) = f_{12}(\lambda;\, \mathcal{R}_n \xi_{1n}, \ldots) = \mathcal{R}_n f_{12}(\lambda;\, \xi_{1n}, \ldots)$$

etc, i.e. the equivariance under $\hat{S}_\mathrm{H}$. The argument is identical for 'vertical' transformations.

The main consequence of this lemma is the following.

Theorem 3. Consider a planar lattice satisfying the assumptions of lemma 2. Then, once a solution $\xi_0(t)$ has been found for the reduced two-dimensional problem, there are for the lattice at least four independent periodic solutions, given by

$$\begin{aligned} &\xi_{\alpha\beta}(t) = \xi_0(t) \\ &\xi_{\alpha\beta}(t) = [\mathcal{R}_n]^\beta \xi_0(t) = \mathcal{R}(2\pi\beta/n)\xi_0(t) \\ &\xi_{\alpha\beta}(t) = [\mathcal{R}_m]^\alpha \xi_0(t) = \mathcal{R}(2\pi\alpha/m)\xi_0(t) \\ &\xi_{\alpha\beta}(t) = [\mathcal{R}_m]^\alpha [\mathcal{R}_n]^\beta \xi_0(t) = \mathcal{R}(2\pi(\alpha/m + \beta/n))\xi_0(t). \end{aligned}$$

The proof of this result follows easily from (15), combining the covariance under $\hat{S}_H$, $\hat{S}_V$ and $\hat{H}_0$, $\hat{V}_0$. Note the structure of a 'wave' travelling horizontally (vertically) shown by the second (third) solution, and of a 'planar' wave shown by the last one. Clearly in the case of a one-dimensional lattice, one finds in this way two independent solutions.

In order to illustrate the above result for a planar lattice, let us consider explicitly an example, with a (relatively) few number of components. Let $N=12$, $m=2$, $n=3$, and let

$$\dot{\xi}_{11}=g(\lambda;\xi_{12})+h(\lambda;\xi_{22}) \qquad \dot{\xi}_{12}=g(\lambda;\xi_{13})+h(\lambda;\xi_{23})$$
$$\dot{\xi}_{13}=g(\lambda;\xi_{11})+h(\lambda;\xi_{21}) \qquad \dot{\xi}_{21}=g(\lambda;\xi_{22})+h(\lambda;\xi_{12})$$
$$\dot{\xi}_{22}=g(\lambda;\xi_{23})+h(\lambda;\xi_{13}) \qquad \dot{\xi}_{23}=g(\lambda;\xi_{21})+h(\lambda;\xi_{11})$$

where g, h are arbitrary SO(2)-covariant functions.

It is easy to verify that this system satisfies all the above assumptions. Choosing, for instance,

$$g(\lambda,x)=\lambda x+|x|^2Jx \qquad h(\lambda,x)=Jx+|x|^2x$$

where $x\in\mathbb{R}^2$, $\lambda\in\mathbb{R}$, and $J=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, standard Hopf conditions for the existence of a periodic bifurcating solution are verified for the elementary two-dimensional site, and the four solutions given by theorem 3 are respectively:

$$\xi_{11}=\xi_{12}=\ldots=\xi_{23}=r\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix} \tag{i}$$
$$\lambda=-r^2 \qquad \omega=1+r^2$$

$$\xi_{11}=\xi_{21}=r\begin{pmatrix}\cos\omega t\\ -\sin\omega t\end{pmatrix}$$
$$\xi_{12}=\xi_{22}=r\begin{pmatrix}\cos(\omega t+2\pi/3)\\ \sin(-\omega t+2\pi/3)\end{pmatrix} \qquad \xi_{13}=\xi_{23}=r\begin{pmatrix}\cos(\omega t+4\pi/3)\\ \sin(-\omega t+4\pi/3)\end{pmatrix} \tag{ii}$$
$$\lambda=-\sqrt{3}-r^2(1+\sqrt{3}) \qquad \omega=2+r^2(1+\sqrt{3})/2$$

$$\xi_{11}=\xi_{12}=\xi_{13}=-\xi_{21}=-\xi_{22}=-\xi_{23}=r\begin{pmatrix}\cos\omega t\\ -\sin\omega t\end{pmatrix} \tag{iii}$$
$$\lambda=r^2 \qquad \omega=1-r^2$$

$$\xi_{11}=-\xi_{21}=r\begin{pmatrix}\cos\omega t\\ -\sin\omega t\end{pmatrix}$$
$$\xi_{12}=-\xi_{22}=r\begin{pmatrix}\cos(\omega t+2\pi/3)\\ \sin(-\omega t+2\pi/3)\end{pmatrix} \qquad \xi_{13}=-\xi_{23}=r\begin{pmatrix}\cos(\omega t+4\pi/3)\\ \sin(-\omega t+4\pi/3)\end{pmatrix} \tag{iv}$$
$$\lambda=-\sqrt{3}-2r^2 \qquad \omega=1+r^2(1+\sqrt{3})/2.$$

In this letter, we have intentionally confined our attention to the general aspects of the problem, without any intention of specific applications, which in fact would deserve a separate and adequate consideration. Let us just mention that the above scheme might be suitably applied, for instance, to spin waves, and in particular to

Heisenberg-type models with non-linear spin–spin and spin–magnetic field interactions (as regards the SO(2) symmetric case); it might be applied as well to waves in crystals, in which case the presence of an external constant magnetic field would break the global SO(3) symmetry down to SO(2).

A final remark, which could be especially relevant in these applications, is the following. Even if the problem is non-linear, it is known [8] that, under precise hypotheses (essentially, some rather general symmetry property, e.g. SO(2) symmetry, and the existence of asymptotic stable solutions), the asymptotic solutions behave precisely as solutions of a linear equation: a consequence of this fact is that linear superpositions of different waves are allowed in this way.

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